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A COLLOCATION METHOD OVER STAGGERED GRIDS FOR THE STOKES PROBLEM

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SUMMARY

This paper is devoted to the description and the detailed numerical analysis of a new spectral collocation method for the Stokes problem in a square, involving three staggered grids.

KEY WORDS Spectral methods Stokes problem Collocation Staggered grids

1. STAGGERED GRIDS: AN INTRODUCTION

We are interested in the approximation of the following Stokes equations over the square $\Omega = \Lambda^2$, with $\Lambda =]-1, 1[$: *Find a velocity* $\mathbf{u} = (u, v)$ *and a pressure* p *such that*

$$-\nu\,\Delta u + \partial p/\partial x = f \quad \text{in } \Omega, \tag{1a}$$

$$-\nu\,\Delta v + \partial p/\partial y = g \quad \text{in } \Omega, \tag{1b}$$

$$\operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega, \tag{2}$$

where the body forces $\mathbf{f} = (f, g)$ are given and ν is a positive real number. This system is provided with homogeneous Dirichlet boundary conditions

$$\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega. \tag{3}$$

Let $N \geqslant 1$ be a fixed integer. We want to use a collocation spectral method, hence we search for approximations $\mathbf{u}_N$ and p_N of $\mathbf{u}$ and p such that each component of $\mathbf{u}_N$ and p_N belong to the space $\mathbb{P}_n(\bar{\Omega})$ of all polynomials of degree $\leqslant n$ with respect to each variable, where n is close to N. The discrete solution will be chosen in order to satisfy the Stokes equations at some points in $\bar{\Omega}$ called collocation points. As pointed out by Orszag[1] and Gottlieb,[2] these points are related to the Gauss-type quadrature formulae associated with the Legendre orthogonal polynomials (we refer to Davis and Rabinowitz[3] for details about numerical integration). A general presentation of spectral methods can be found in Gottlieb and Orszag[4] or Canuto *et al.*[5]

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This paper is motivated by the difficulty encountered in most numerical schemes due to the presence of spurious modes for the pressure; these modes pollute the solution and sometimes lead to incoherent results. The difficulty is connected with the well-known problem of finding a compatible choice of discrete spaces for the velocity and the pressure. In Bernardi *et al.*[6–8] we have explained the theoretical properties that these spaces must satisfy. They have been initially pointed out by Brezzi[9] as the inf–sup condition.

In a first method, introduced by Morchoisne[10] and generalized by Métivet,[11] a collocation grid Ξ_N is defined by the tensor product of the nodes of a Gauss–Lobatto integration rule using $N+1$ points; the discrete velocity $\mathbf{u}_N$ and the discrete pressure p_N are respectively chosen in $\mathbb{P}_N(\bar{\Omega})^2 \cap \mathbf{H}_0^1(\Omega)^2$ and $\mathbb{P}_N(\bar{\Omega})$ so as to satisfy equations (1) at each point of $\Xi_N \cap \Omega$ and equation (2) at each point of Ξ_N. In this method the equations are not mutually independent and the pressure is not uniquely determined by the equations; indeed the solution consists of an affine subspace of dimension eight in $\mathbb{P}_N(\bar{\Omega})$. However, it has been proved[7,8] and tested in numerical experiments[10,11] that such a scheme (after elimination of the spurious modes and the redundant equations) provides a good approximation of the velocity while a post-treatment is necessary to obtain a good discrete pressure. Nevertheless, the numerical difficulties encountered justify the research of a method where no spurious mode comes to pollute the solution.

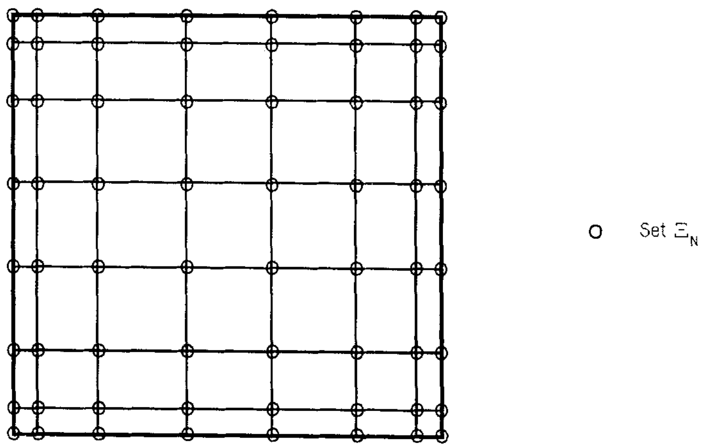

Figure 1. Using a unique collocation grid

When the problem of spurious modes arises in a finite difference context, a well-known tool to solve it consists of the introduction of staggered grids, i.e., different grids on which the various equations (1) and (2) are satisfied. This suggests the idea of a similar strategy for spectral methods. A first attempt in this direction can be found in Zang and Hussaini[12] and analysed for the approximation of the Navier–Stokes equations with periodic boundary conditions in every direction but one.[6] Another attempt due to Montigny–Rannou and Morchoisne,[13] in the more interesting case of non-periodic boundary conditions, has drastically reduced the number of spurious modes. In this method two grids are considered, one for the velocity and another for the pressure. More precisely, the grid Ξ_N is the same as in the previous method (see Figure 1) and a new grid Ξ'_N is defined by the tensor product of the nodes of a Gauss integration rule using N points. From the properties of the Gauss and Gauss–Lobatto points, these two grids are staggered. The numerical velocity and pressure are then searched in $\mathbb{P}_N(\bar{\Omega})^2 \cap \mathbf{H}_0^1(\Omega)^2$ and $\mathbb{P}_{N-1}(\bar{\Omega})$ respectively in order to satisfy equations (1) at each point of $\Xi_N \cap \Omega$ and equation (2) at each point of Ξ'_N. Unfortunately the algorithm

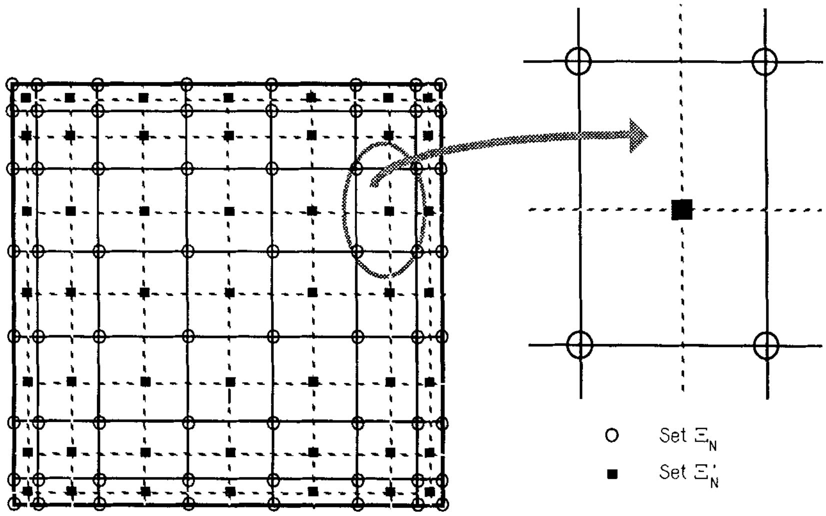

Figure 2. Using two collocation grids

does not solve the main difficulty since, besides the constant mode, there exists one further spurious mode, namely the function $L'_N(x)L'_N(y)$, where L_N is the polynomial of degree N in the family of Legendre polynomials. *A priori* it may seem easy to filter this unique mode, but we must point out that, in the three-dimensional case of a cubic domain, the number of spurious modes is equal to $3N$-4 in this last method.

The aim of this work is to propose and analyse a new collocation method where no spurious mode is present. The discrete pressure is again sought in $\mathbb{P}_{N-1}(\bar{\Omega})$, while we must choose a less natural space of discrete velocities. Equation (2) is again satisfied on the nodes of the same grid Ξ'_N as in Figure 2, but two different grids are introduced for equations (1a) and (1b) respectively; each one is staggered with respect to Ξ'_N in a different direction. The relative position of these three grids is the spectral analogue of a situation which is well known in a finite difference context.[14] Moreover, this algorithm actually presents numerical properties which will appear later on and which compensate the complexity of using three grids.

In Section 2 we shall give a description of the method and indicate a suitable variational formulation of it. In Sections 3 and 4 we shall prove some properties of the associated discrete Laplace operator and discrete pressure gradient respectively. These results will be used in Section 5 to derive error estimates for the velocity and the pressure.

2. DESCRIPTION OF THE ALGORITHM

In the sequel we denote by $(L_n)_{n\in\mathbb{N}}$ the family of Legendre orthogonal polynomials on the interval $\Lambda =]-1, 1[$, where L_n is the polynomial of degree n normalized by the condition $L_n(\pm 1) = (\pm 1)^n$. We recall that $\mathbb{P}_n(\bar{\Lambda})$, $n\in\mathbb{N}$, is the space of polynomials of degree $\leqslant n$ restricted to the interval $\bar{\Lambda}$. Let us introduce the set $\{\zeta_1, \ldots, \zeta_N\}$ of the nodes of the Gauss integration formula with N points over Λ, with $\zeta_1 < \cdots < \zeta_N$; it is well known (Section 2.7 of Reference 3) that these points are the N zeros of L_N. Let us also introduce the set $\{\xi_0, \ldots, \xi_N\}$ of the nodes of the Gauss–Lobatto integration formula with $N+1$ points over $\bar{\Lambda}$, with $-1 = \xi_0 < \xi_1 < \cdots < \xi_N = 1$; these points are the $N+1$ zeros of $(1-\zeta^2)L'_N$. It is an easy matter to note that, for any i, $1 \leqslant i \leqslant N$, we have $\xi_{i-1} \leqslant \zeta_i \leqslant \xi_i$. We recall (Section 2.7 of Reference 3 and Section 3.2 of Reference 15) that there exist $2N+1$ positive real numbers

$(\omega_1, \ldots, \omega_N, \rho_0, \ldots, \rho_N)$ such that

$$\forall \varphi \in \mathbb{P}_{2N-1}(\bar{\Lambda}), \quad \int_\Lambda \varphi(\zeta)\,\mathrm{d}\zeta = \sum_{i=1}^{N} \varphi(\zeta_i)\omega_i, \tag{4}$$

$$\forall \varphi \in \mathbb{P}_{2N-1}(\bar{\Lambda}), \quad \int_\Lambda \varphi(\zeta)\,\mathrm{d}\zeta = \sum_{i=0}^{N} \varphi(\xi_i)\rho_i, \tag{5}$$

$$\forall \varphi \in \mathbb{P}_{N}(\bar{\Lambda}), \quad \int_\Lambda \varphi(\zeta)L_N(\zeta)\,\mathrm{d}\zeta = (N/(2N+1)) \sum_{i=0}^{N} \varphi(\xi_i)L_N(\xi_i)\rho_i. \tag{6}$$

The tensor products of these two sets of points give us different grids over $\bar{\Omega}$ that will be used for the definition of the collocation approximation of problem (1), (2). Let us consider the grids

$$\Xi_{N,x} = \{(\xi_i, \zeta_j), 0 \leqslant i \leqslant N, 1 \leqslant j \leqslant N\}, \tag{7a}$$

$$\Xi_{N,y} = \{(\zeta_i, \xi_j), 1 \leqslant i \leqslant N, 0 \leqslant j \leqslant N\}, \tag{7b}$$

$$\Xi'_N = \{(\zeta_i, \zeta_j), 1 \leqslant i,j \leqslant N\}. \tag{8}$$

Due to the relative position of the Gauss and Gauss–Lobatto points, $\Xi_{N,x}$ is staggered with respect to Ξ'_N in the x-direction and $\Xi_{N,y}$ is staggered with respect to Ξ'_N in the y-direction.

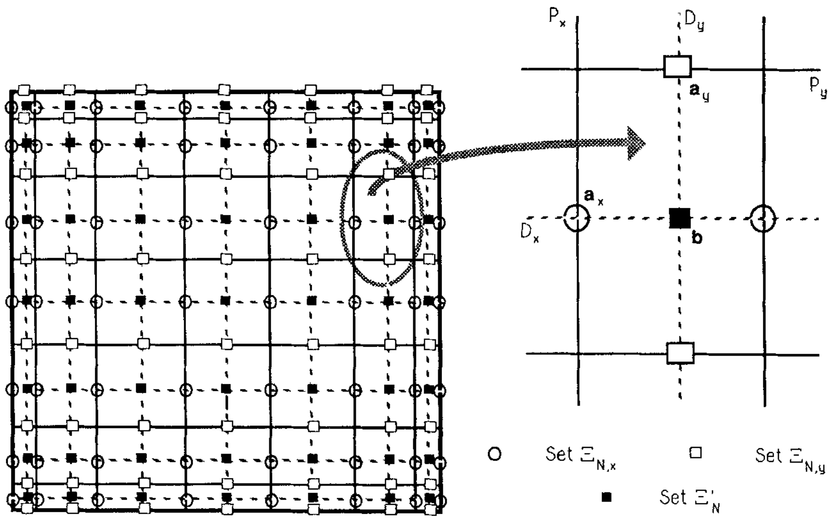

Figure 3. The three collocation grids (for $N = 7$)

The use of different grids implies that the spaces of discretization for the components of the velocity and the pressure will not coincide. For any non-negative integer n we set $\mathbb{P}_n(\bar{\Omega}) = \mathbb{P}_n(\bar{\Lambda}) \otimes \mathbb{P}_n(\bar{\Lambda})$. We first introduce the discrete space $\mathbf{X}_N = \mathbf{X}_{N,x} \times \mathbf{X}_{N,y}$ defined by

$$\mathbf{X}_{N,x} = [\mathbb{P}_N(\bar{\Lambda}) \otimes \mathbb{P}_{N+1}(\bar{\Lambda})] \cap \mathbf{H}_0^1(\Omega), \tag{9a}$$

$$\mathbf{X}_{N,y} = [\mathbb{P}_{N+1}(\bar{\Lambda}) \otimes \mathbb{P}_N(\bar{\Lambda})] \cap \mathbf{H}_0^1(\Omega). \tag{9b}$$

In order to obtain a unique solution for the pressure, we take the constant mode off the space of discrete pressures; so we define

$$\mathbf{M}_N = \left\{ q \in \mathbb{P}_{N-1}(\bar{\Omega});\ \int_\Omega q(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0 \right\}. \tag{10}$$

Our discrete problem is: *Find a discrete solution* $\mathbf{u}_N = (u_N, v_N)$ *in* $\mathbf{X}_{N,x} \times \mathbf{X}_{N,y}$ *and a discrete pressure* p_N *in* $\mathbf{M}_N$ *such that*

$$\forall \mathbf{x} \in \Xi_{N,x} \cap \Omega,\quad [-\nu\,\Delta u_N + \partial p_N/\partial x](\mathbf{x}) = f(\mathbf{x}), \tag{11a}$$

$$\forall \mathbf{x} \in \Xi_{N,y} \cap \Omega,\quad [-\nu\,\Delta v_N + \partial p_N/\partial y](\mathbf{x}) = g(\mathbf{x}), \tag{11b}$$

$$\forall \mathbf{x} \in \Xi'_N,\ (\operatorname{div} \mathbf{u})(\mathbf{x}) = 0. \tag{12}$$

Remark 1

Of course it is an easy matter to define the same algorithm by using the Chebyshev approximation instead of the Legendre one: indeed, it suffices to replace the ζ_i by the zeros of T_N and the ξ_i by the zeros of $(1-\zeta^2)T'_N$, where $T_N(\zeta) = \cos(N\cos^{-1}\zeta)$ denotes the Chebyshev polynomial of degree N. From a numerical point of view[4,11,13] this last choice has to be preferred, since it is possible in this case to use a fast Fourier transform algorithm to decrease the computation time. But the analysis is then much more difficult to perform, since the Chebyshev polynomials are orthogonal with respect to the weighted measure $(1-\zeta^2)^{-1/2}\,\mathrm{d}\zeta$ and the gradient and divergence operators are no more adjoint to each other for this measure. However, it is now under consideration, and we think that it can be achieved thanks to the techniques developed for a single-grid algorithm.[8] The theoretical results are foreseen to be the same as in the Legendre case.

Remark 2

It is straightforward to extend the algorithm to the three-dimensional case by using four staggered grids. To approximate the Stokes problem in the cubic domain $]-1,1[^3$, we define the grids

$$\Xi_{N,x} = \{(\xi_i, \zeta_j, \zeta_k), 0 \leqslant i \leqslant N, 1 \leqslant j,k \leqslant N\},$$
$$\Xi_{N,y} = \{(\zeta_i, \xi_j, \zeta_k), 0 \leqslant j \leqslant N, 1 \leqslant i,k \leqslant N\},$$
$$\Xi_{N,z} = \{(\zeta_i, \zeta_j, \xi_k), 0 \leqslant k \leqslant N, 1 \leqslant i,j \leqslant N\},$$
$$\Xi'_N = \{(\zeta_i, \zeta_j, \zeta_k), 1 \leqslant i,j,k \leqslant N\}.$$

Next, we seek a discrete velocity $\mathbf{u}_N$ in $\mathbf{X}_N = \mathbf{X}_{N,x} \times \mathbf{X}_{N,y} \times \mathbf{X}_{N,z}$, with

$$\mathbf{X}_{N,x} = [\mathbb{P}_N(\bar{\Lambda}) \otimes \mathbb{P}_{N+1}(\bar{\Lambda}) \otimes \mathbb{P}_{N+1}(\bar{\Lambda})] \cap \mathbf{H}^1_0(\Lambda^3),$$
$$\mathbf{X}_{N,y} = [\mathbb{P}_{N+1}(\bar{\Lambda}) \otimes \mathbb{P}_N(\bar{\Lambda}) \otimes \mathbb{P}_{N+1}(\bar{\Lambda})] \cap \mathbf{H}^1_0(\Lambda^3),$$
$$\mathbf{X}_{N,z} = [\mathbb{P}_{N+1}(\bar{\Lambda}) \otimes \mathbb{P}_{N+1}(\bar{\Lambda}) \otimes \mathbb{P}_N(\bar{\Lambda})] \cap \mathbf{H}^1_0(\Lambda^3),$$

and a discrete pressure p_N in $\mathbf{M}_N = \{q \in \mathbb{P}_{N-1}(\bar{\Lambda}^3); \int_{\Lambda^3} q(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0\}$ such that the four equations are satisfied at the nodes of $\Xi_{N,x} \cap \Omega$, $\Xi_{N,y} \cap \Omega$, $\Xi_{N,z} \cap \Omega$ and Ξ'_N respectively. It is lengthy to write but not more difficult to implement than in the two-dimensional case. Moreover, the numerical analysis turns out to be the same one and leads to similar results.

Remark 3

As we shall see in Section 4, this algorithm has no spurious modes; we want to point out another reason why we think that this discretization is better than other ones. We can say that the grids are staggered in the 'good' directions. Indeed, let us consider the partial derivatives which appear in equation (11a): with the notation of Figure 3, the computation of Δu at a point $\mathbf{a}_x$ by a pseudospectral method involves the values of u at any node of the straight line D_x (resp. P_x) in order to calculate $\partial^2 u/\partial x^2$ (resp. $\partial^2 u/\partial y^2$); in the same way, to evaluate the value of $\partial p/\partial x$ at $\mathbf{a}_x$, we need only to know the values of p at any node of the straight line D_x. Using the same kind of argument for equations (11b) and (12), we check that the verification of our discrete problem only requires a one-dimensional derivation process of interpolation, in contrast to other algorithms.[13]

Remark 4

Using the notion of Lagrange interpolants will provide us with suitable bases for the various discrete spaces. More precisely, let us define the polynomials q_i, $1 \leqslant i \leqslant N$, in $\mathbb{P}_{N-1}(\bar{\Lambda})$ by

$$\forall i, \forall j, 1 \leqslant i,j \leqslant N, \quad q_i(\zeta_j) = \delta_{ij}, \tag{13}$$

and the polynomials r_i, $0 \leqslant i \leqslant N$, in $\mathbb{P}_N(\bar{\Lambda})$ by

$$\forall i, \forall j, 0 \leqslant i,j \leqslant N, \quad r_i(\xi_j) = \delta_{ij}, \tag{14}$$

where δ_{ij} stands for the Kronecker symbol. It is now an easy matter to check that the set

$$\{Q^x_{ij}(x,y) = r_i(x)(1-y^2)q_j(y)\}_{1 \leqslant i \leqslant N-1, 1 \leqslant j \leqslant N}$$

is a basis of $\mathbf{X}_{N,x}$, that similarly the set

$$\{Q^y_{ij}(x,y) = (1-x^2)q_i(x)r_j(y)\}_{1 \leqslant i \leqslant N, 1 \leqslant j \leqslant N-1}$$

is a basis of $\mathbf{X}_{N,y}$ and finally that the set

$$\{Q'_{ij}(x,y) = q_i(x)q_j(y)\}_{1 \leqslant i,j \leqslant N}$$

is a basis of $\mathbb{P}_{N-1}(\bar{\Omega})$.

In order to analyse the discrete problem, we need a variational formulation of it. First, let us recall some basic results concerning the continuous problem. We define the two spaces where the solution $(\mathbf{u}, p)$ has to be searched. For the velocity we set

$$\mathbf{X} = [\mathbf{H}^1_0(\Omega)]^2, \tag{15}$$

while for the pressure we choose

$$\mathbf{M} = \mathbf{L}^2_0(\Omega) = \left\{ q \in \mathbf{L}^2(\Omega); \int_\Omega q(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0 \right\}. \tag{16}$$

We then define the two bilinear forms

$$\forall \mathbf{u} = (u,v) \in \mathbf{X}, \forall \mathbf{w} = (w,z) \in \mathbf{X}, \quad a(\mathbf{u},\mathbf{w}) = \nu \int_\Omega \nabla\mathbf{u}(\mathbf{x}) \nabla\mathbf{w}(\mathbf{x})\,\mathrm{d}\mathbf{x}, \tag{17}$$

$$\forall \mathbf{u} = (u,v) \in \mathbf{X}, \forall q \in \mathbf{M}, \quad b(\mathbf{u},q) = -\int_\Omega (\operatorname{div}\mathbf{u})(\mathbf{x})q(\mathbf{x})\,\mathrm{d}\mathbf{x}. \tag{18}$$

It is standard to check that, whenever the body forces $\mathbf{f}$ are in the dual space $\mathbf{H}^{-1}(\Omega)^2$ of $\mathbf{X}$, problem (1)–(3) has the following equivalent variational formulation: *Find* $\mathbf{u}=(u,v)$ *in* $\mathbf{X}$ *and* p *in* $\mathbf{M}$ *such that*

$$\begin{aligned} &\forall \mathbf{w}\in\mathbf{X}, \quad a(\mathbf{u},\mathbf{w})+b(\mathbf{w},p)=(\mathbf{f},\mathbf{w}),\\ &\forall q\in\mathbf{M}, \quad b(\mathbf{u},q)=0, \end{aligned} \tag{19}$$

where $(\cdot,\cdot)$ denotes the scalar product in $\mathbf{L}^2(\Omega)^2$ as well as the duality pairing between $\mathbf{H}^{-1}(\Omega)^2$ and $\mathbf{H}^1_0(\Omega)^2$.

The formulation of the discrete problem leads us to introduce the two bilinear forms a_N,

$$\forall \mathbf{u}=(u,v)\in\mathscr{C}^2(\bar\Omega)^2, \forall \mathbf{w}=(w,z)\in\mathscr{C}^0(\bar\Omega)^2,$$

$$\begin{aligned} a_N(\mathbf{u},\mathbf{w}) = &-\nu\sum_{i=0}^{N}\sum_{j=1}^{N}\Delta u(\xi_i,\zeta_j)w(\xi_i,\zeta_j)\rho_i\omega_j\\ &-\nu\sum_{i=1}^{N}\sum_{j=0}^{N}\Delta v(\zeta_i,\xi_j)z(\zeta_i,\xi_j)\omega_i\rho_j, \end{aligned} \tag{20}$$

and b_N,

$$\forall \mathbf{u}=(u,v)\in\mathscr{C}^0(\bar\Omega)^2, \forall q\in\mathscr{C}^1(\bar\Omega),$$

$$\begin{aligned} b_N(\mathbf{u},q) = &\sum_{i=0}^{N}\sum_{j=1}^{N}u(\xi_i,\zeta_j)(\partial q/\partial x)(\xi_i,\zeta_j)\rho_i\omega_j\\ &+\sum_{i=1}^{N}\sum_{j=0}^{N}v(\zeta_i,\xi_j)(\partial q/\partial y)(\zeta_i,\xi_j)\omega_i\rho_j. \end{aligned} \tag{21}$$

Using (4) and (5), we obtain the important property

$$\forall \mathbf{u}=(u,v)\in\mathbf{X}, \forall q\in\mathbf{M},$$

$$\begin{aligned} b_N(\mathbf{u},q) = &\int_\Lambda\sum_{j=1}^{N}u(x,\zeta_j)(\partial q/\partial x)(x,\zeta_j)\omega_j\,\mathrm{d}x\\ &+\int_\Lambda\sum_{i=1}^{N}v(\zeta_i,y)(\partial q/\partial y)(\zeta_i,y)\omega_i\,\mathrm{d}y, \end{aligned} \tag{22}$$

so that

$$\forall \mathbf{u}\in\mathbf{X}, \forall q\in\mathbf{M}, \quad b_N(\mathbf{u},q) = -\sum_{i=1}^{N}\sum_{j=1}^{N}(\operatorname{div}\mathbf{u})(\zeta_i,\zeta_j)q(\zeta_i,\zeta_j)\omega_i\omega_j. \tag{23}$$

Finally, we define a discrete scalar product $(\cdot,\cdot)_N$ by

$$\forall \mathbf{f}=(f,g)\in\mathscr{C}^0(\bar\Omega)^2, \forall \mathbf{w}=(w,z)\in\mathscr{C}^0(\bar\Omega)^2,$$

$$\begin{aligned} (\mathbf{f},\mathbf{w})_N = &\sum_{i=0}^{N}\sum_{j=1}^{N}f(\xi_i,\zeta_j)w(\xi_i,\zeta_j)\rho_i\omega_j\\ &+\sum_{i=1}^{N}\sum_{j=0}^{N}g(\zeta_i,\xi_j)z(\zeta_i,\xi_j)\omega_i\rho_j. \end{aligned} \tag{24}$$

We are in a position to prove:

Proposition 1

For any $\mathbf{f}$ *in* $\mathscr{C}^0(\bar\Omega)^2$, *the discrete problem (11), (12) is equivalent to the following variational* one: *Find* $\mathbf{u}_N = (u_N, v_N)$ *in* $\mathbf{X}_{N,x} \times \mathbf{X}_{N,y}$ *and* p_N *in* $\mathbf{M}_N$ *such that*

$$\begin{aligned} &\forall \mathbf{w}_N \in \mathbf{X}_N, \quad a_N(\mathbf{u}_N, \mathbf{w}_N) + b_N(\mathbf{w}_N, p_N) = (\mathbf{f}, \mathbf{w}_N)_N, \\ &\forall q_N \in \mathbf{M}_N, \quad b_N(\mathbf{u}_N, q_N) = 0. \end{aligned} \tag{25}$$

Proof. It is an easy matter to check that the discrete problem (11), (12) is equivalent to: *Find* $\mathbf{u}_N = (u_N, v_N)$ *in* $\mathbf{X}_{N,x} \times \mathbf{X}_{N,y}$ *and* p_N *in* $\mathbf{M}_N$ *such that*

$$\forall \mathbf{x} \in \Xi_{N,x} \cap \Omega, \quad [(-\nu \Delta u_N + \partial p_N/\partial x) Q_{ij}^x](\mathbf{x}) = (f Q_{ij}^x)(\mathbf{x}), \quad 1 \leqslant i \leqslant N-1, 1 \leqslant j \leqslant N,$$

$$\forall \mathbf{x} \in \Xi_{N,y} \cap \Omega, \quad [(-\nu \Delta v_N + \partial p_N/\partial y) Q_{ij}^y](\mathbf{x}) = (g Q_{ij}^y)(\mathbf{x}), \quad 1 \leqslant i \leqslant N, 1 \leqslant j \leqslant N-1,$$

$$\forall \mathbf{x} \in \Xi'_N, \quad [(\operatorname{div} \mathbf{u}) Q'_{ij}](\mathbf{x}) = 0, \quad 1 \leqslant i,j \leqslant N,$$

where the polynomials Q_{ij}^x, Q_{ij}^y and Q'_{ij} have been defined in Remark 4. Hence definitions (20), (21), (24) and property (23) lead to the equivalence between problem (11), (12) and the following: *Find* $\mathbf{u}_N = (u_N, v_N)$ *in* $\mathbf{X}_{N,x} \times \mathbf{X}_{N,y}$ *and* p_N *in* $\mathbf{M}_N$ *such that*

$$\begin{aligned} &\forall \mathbf{w}_N \in \mathbf{X}_N, \quad a_N(\mathbf{u}_N, \mathbf{w}_N) + b_N(\mathbf{w}_N, p_N) = (\mathbf{f}, \mathbf{w}_N)_N, \\ &\forall q_N \in \mathbb{P}_{N-1}(\bar{\Omega}), \quad b_N(\mathbf{u}_N, q_N) = 0. \end{aligned} \tag{26}$$

Since we have $\mathbf{M}_N \oplus \mathbb{R} = \mathbb{P}_{N-1}(\bar{\Omega})$, the equivalence between this last problem and (25) is a consequence of the following simple equality (recall (4) and (23)).

$$\forall \mathbf{w}_N \in \mathbf{X}_N, \quad b_N(\mathbf{w}_N, 1) = -\int_\Omega \operatorname{div} \mathbf{w}_N \, d\mathbf{x} = 0.$$

As a matter of fact, if we compare the numbers of unknowns and equations in problem (11), (12), it appears that (12) is over-specified and that numerically we have to impose this divergence-free condition at only any $N^2 - 1$ points among the N^2 of Ξ'_N. The corresponding equation we have not checked is then automatically verified since, due to the ever-present property

$$\sum_{i=1}^N \sum_{j=1}^N (\operatorname{div} \mathbf{u}_N)(\zeta_i, \zeta_j)\omega_i\omega_j = \int_\Omega \operatorname{div} \mathbf{u}_N \, d\mathbf{x} = 0,$$

there exists one (and only one, as will be proved later) linear dependency condition between the values of $\operatorname{div} \mathbf{u}_N$ on the set Ξ'_N.

3. PROPERTIES OF THE DISCRETE LAPLACE OPERATOR

In this section we study the properties of the bilinear form a_N defined in (20). This form is related to the discretization of a vectorial Laplace operator involving two staggered grids.

First, we need some properties of the quadrature formulae.

Lemma 1

For any φ *in* $\mathbb{P}_{N-1}(\bar{\Lambda})$, *the following inequalities hold*:

$$\int_\Lambda (1-\zeta^2)\varphi(\zeta)^2 \, d\zeta \leqslant \sum_{i=1}^N (1-\zeta_i^2)\varphi(\zeta_i)^2\omega_i \leqslant 2\int_\Lambda (1-\zeta^2)\varphi(\zeta)^2 \, d\zeta. \tag{27}$$

Proof. Let us write φ in the basis $(L'_n)_{1\leqslant n\leqslant N}$:

$$\varphi(\zeta)=\sum_{n=1}^{N}\varphi_n L'_n(\zeta).$$

Thanks to the formula (Section 1.13 of Reference 3)

$$(\mathrm{d}/\mathrm{d}\zeta)((1-\zeta^2)L'_n)+n(n+1)L_n=0 \tag{28}$$

and using $\|L_n\|^2_{0,\Lambda}=1/(n+1/2)$, we know that

$$\int_\Lambda (1-\zeta^2)\varphi(\zeta)^2\,\mathrm{d}\zeta=\sum_{n=1}^{N}\varphi_n^2 n(n+1)/(n+1/2). \tag{29}$$

On the other hand, due to (4), we also have

$$\sum_{i=1}^{N}(1-\zeta_i^2)\varphi(\zeta_i)^2\omega_i=\sum_{n=1}^{N-1}\varphi_n^2 n(n+1)/(n+1/2)+\varphi_N^2\sum_{i=1}^{N}(1-\zeta_i^2)L'_N(\zeta_i)^2\omega_i. \tag{30}$$

To study this last term, we recall the formulae (Section 1.13 of Reference 3)

$$\int L_n(\zeta)\,\mathrm{d}\zeta=(1/(2n+1))(L_{n+1}-L_{n-1}), \tag{31}$$

$$(n+1)L_{n+1}=(2n+1)\zeta L_n-nL_{n-1}. \tag{32}$$

From (28) we see that $(1-\zeta^2)L'_N(\zeta)$ is the primitive function of $-N(N+1)L_N$ which vanishes for $\zeta=\pm1$, hence is equal to $-[N(N+1)/(2N+1)](L_{N+1}-L_{N-1})$ by (31). Since the ζ_i, $1\leqslant i\leqslant N$, are the zeros of L_N, using (32), we compute

$$(1-\zeta_j^2)L'_N(\zeta_j)=-[N(N+1)/(2N+1)][-(N/(N+1))L_{N-1}(\zeta_j)-L_{N-1}(\zeta_j)]=NL_{N-1}(\zeta_j),$$

which gives

$$\sum_{i=1}^{N}(1-\zeta_i^2)L'_N(\zeta_i)^2\omega_i=N\sum_{i=1}^{N}L_{N-1}(\zeta_i)L'_N(\zeta_i)\omega_i=N\int_\Lambda L_{N-1}(\zeta)L'_N(\zeta)\,\mathrm{d}\zeta$$

$$=N\Big([L_{N-1}L_N]^1_{-1}-\int_\Lambda L'_{N-1}(\zeta)L_N(\zeta)\,\mathrm{d}\zeta\Big)=2N.$$

From this last formula, together with (29) and (30), we deduce the lemma.

Lemma 2

For any φ in $\mathbb{P}_{N-1}(\bar\Lambda)$, the following inequalities hold:

$$cN^{-1}\int_\Lambda(1-\zeta^2)^2\varphi(\zeta)^2\,\mathrm{d}\zeta\leqslant\sum_{i=1}^{N}(1-\zeta_i^2)^2\varphi(\zeta_i)^2\omega_i\leqslant c'\int_\Lambda(1-\zeta^2)^2\varphi(\zeta)^2\,\mathrm{d}\zeta. \tag{33}$$

Proof. Let us now write φ in the basis $(L''_n)_{2\leqslant n\leqslant N+1}$:

$$\varphi(\zeta)=\sum_{n=2}^{N+1}\tilde\varphi_n L''_n(\zeta).$$

Derivating the formula (28), we obtain

$$(\mathrm{d}/\mathrm{d}\zeta)((1-\zeta^2)L''_n)=(\mathrm{d}/\mathrm{d}\zeta)(2\zeta L'_n-n(n+1)L_n)=2\zeta L''_n-(n-1)(n+2)L'_n,$$

whence

$$(\mathrm{d}/\mathrm{d}\zeta)((1-\zeta^2)^2 L_n'') = (1-\zeta^2)(2\zeta L_n'' - (n-1)(n+2)L_n') - 2\zeta(1-\zeta^2)L_n'',$$

so that using (28) once more gives

$$(\mathrm{d}^2/\mathrm{d}\zeta^2)((1-\zeta^2)^2 L_n'') = (n-1)n(n+1)(n+2)L_n. \tag{34}$$

This formula yields

$$\int_\Lambda (1-\zeta^2)^2 \varphi(\zeta)^2\,\mathrm{d}\zeta = \sum_{n=2}^{N+1} \tilde{\varphi}_n^2 (n-1)n(n+1)(n+2)/(n+1/2). \tag{35}$$

On the other hand, due to (4), we also have

$$\begin{aligned}\sum_{i=1}^{N} &(1-\zeta_i^2)^2 \varphi(\zeta_i)^2 \omega_i \\ &= \sum_{n=2}^{N-2} \tilde{\varphi}_n^2 (n-1)n(n+1)(n+2)/(n+1/2) + \tilde{\varphi}_N^2 \sum_{i=1}^{N} (1-\zeta_i^2)^2 L_N''(\zeta_i)^2 \omega_i \\ &\quad + \sum_{i=1}^{N} (1-\zeta_i^2)^2 [\tilde{\varphi}_{N-1} L_{N-1}''(\zeta_i) + \tilde{\varphi}_{N+1} L_{N+1}''(\zeta_i)]^2 \omega_i.\end{aligned} \tag{36}$$

We must compute the two last terms.

(1) Since the ζ_i, $1 \leqslant i \leqslant N$, are the zeros of L_N, using successively (28), (32) and (31), we have

$$\begin{aligned}(1-\zeta_i^2)L_N''(\zeta_i) &= 2\zeta_i L_N'(\zeta_i) = 2(\zeta L_N)'(\zeta_i) = (2/(2N+1))((N+1)L_{N+1}' + N L_{N-1}')(\zeta_i) \\ &= 2L_{N-1}'(\zeta_i),\end{aligned}$$

so that

$$\begin{aligned}\sum_{i=1}^{N} (1-\zeta_i^2)^2 L_N''(\zeta_i)^2 \omega_i &= 4\sum_{i=1}^{N} L_{N-1}'(\zeta_i)^2 \omega_i = 4\int_\Lambda L_{N-1}'(\zeta)^2\,\mathrm{d}\zeta \\ &= 4\left([L_{N-1}L_{N-1}']_{-1}^{1} - \int_\Lambda L_{N-1}(\zeta) L_{N-1}''(\zeta)\,\mathrm{d}\zeta\right) \\ &= 4(L_{N-1}(1)L_{N-1}'(1) - L_{N-1}(-1)L_{N-1}'(-1));\end{aligned}$$

using (28) finally gives

$$\sum_{i=1}^{N} (1-\zeta_i^2)^2 L_N''(\zeta_i)^2 \omega_i = 4(N-1)N. \tag{37}$$

(2) Since the ζ_i, $1 \leqslant i \leqslant N$, are the zeros of L_N, by (28), (31) and (32), we have

$$\begin{aligned}(1-\zeta_i^2)L_{N+1}''(\zeta_i) &= 2\zeta_i L_{N+1}'(\zeta_i) - (N+1)(N+2)L_{N+1}(\zeta_i) \\ &= 2\zeta_i L_{N-1}'(\zeta_i) + N(N+2)L_{N-1}(\zeta_i) \\ &= (1-\zeta_i^2)L_{N-1}''(\zeta_i) + N(2N+1)L_{N-1}(\zeta_i),\end{aligned}$$

so that

$$\begin{aligned}\sum_{i=1}^{N} &(1-\zeta_i^2)^2 [\tilde{\varphi}_{N-1} L_{N-1}''(\zeta_i) + \tilde{\varphi}_{N+1} L_{N+1}''(\zeta_i)]^2 \omega_i \\ &= \sum_{i=1}^{N} [(\tilde{\varphi}_{N-1} + \tilde{\varphi}_{N+1})(1-\zeta_i^2)L_{N-1}''(\zeta_i) + N(2N+1)\tilde{\varphi}_{N+1} L_{N-1}(\zeta_i)]^2 \\ &= \int_\Lambda [(\tilde{\varphi}_{N-1} + \tilde{\varphi}_{N+1})(1-\zeta^2)L_{N-1}''(\zeta) + N(2N+1)\tilde{\varphi}_{N+1} L_{N-1}(\zeta)]^2\,\mathrm{d}\zeta.\end{aligned}$$

Using (28), (32) and (33), we compute

$$\int_\Lambda (1-\zeta^2)L_n''(\zeta)L_n(\zeta)\,d\zeta = \int_\Lambda (2\zeta L_n'(\zeta) - n(n+1)L_n(\zeta))L_n(\zeta)\,d\zeta$$
$$= (2/(2n+1))\int_\Lambda L_n'(\zeta)((n+1)L_{n+1}(\zeta) + nL_{n-1}(\zeta))\,d\zeta - n(n+1)/(n+1/2)$$
$$= 4n/(2n+1) - n(n+1)/(n+1/2) = -n(n-1)/(n+1/2).$$

Using also (34), we obtain

$$\sum_{i=1}^{N} (1-\zeta_i^2)^2[\tilde{\varphi}_{N-1}L_{N-1}''(\zeta_i) + \tilde{\varphi}_{N+1}L_{N+1}''(\zeta_i)]^2\omega_i$$
$$= (\tilde{\varphi}_{N-1} + \tilde{\varphi}_{N+1})^2[(N-2)(N-1)N(N+1)/(N-1/2)]$$
$$- 2(\tilde{\varphi}_{N-1} + \tilde{\varphi}_{N+1})\tilde{\varphi}_{N+1}[(N-2)(N-1)N(2N+1)/(N-1/2)]$$
$$+ \tilde{\varphi}_{N+1}^2[N^2(2N+1)^2/(N-1/2)]$$
$$= (N/(N-1/2))[\tilde{\varphi}_{N-1}^2(N-2)(N-1)(N+1) - 2\tilde{\varphi}_{N-1}\tilde{\varphi}_{N+1}(N-2)(N-1)N$$
$$+ \tilde{\varphi}_{N+1}^2(N^3 + 12N^2 - 2N - 2)].$$

The inequality $2|\tilde{\varphi}_{N-1}||\tilde{\varphi}_{N+1}| \leqslant \tilde{\varphi}_{N-1}^2 + \tilde{\varphi}_{N+1}^2$ finally yields, on one hand,

$$\sum_{i=1}^{N} (1-\zeta_i^2)^2[\tilde{\varphi}_{N-1}L_{N-1}''(\zeta_i) + \tilde{\varphi}_{N+1}L_{N+1}''(\zeta_i)]^2\omega_i$$
$$\leqslant \tilde{\varphi}_{N-1}^2((N-2)(N-1)N(2N+1)/(N-1/2))$$
$$+ \tilde{\varphi}_{N+1}^2(N(2N^3 + 9N^2 - 2)/(N-1/2) \tag{38}$$

and, on the other hand,

$$\sum_{i=1}^{N} (1-\zeta_i^2)^2[\tilde{\varphi}_{N-1}L_{N-1}''(\zeta_i) + \tilde{\varphi}_{N+1}L_{N+1}''(\zeta_i)]^2\omega_i \geqslant \tilde{\varphi}_{N-1}^2((N-2)(N-1)N/(N-1/2))$$
$$+ \tilde{\varphi}_{N+1}^2(N(15N^2 - 4N - 2)/(N-1/2)). \tag{39}$$

From (35)–(38) we derive the second inequality of the lemma. From (35)–(37) and (39), noting that the coefficients of $\tilde{\varphi}_{N-1}^2$, $\tilde{\varphi}_N^2$ and $\tilde{\varphi}_{N+1}^2$ are in $\mathcal{O}(N^3)$ in the exact sum and in $\mathcal{O}(N^2)$ in the discrete sum, we obtain the first inequality.

Now we can state some properties of the discrete form a_N.

Proposition 2

The form a_N is uniformly continuous with respect to N, i.e.,

$$\forall \mathbf{u}\in\mathbf{X}_N, \forall \mathbf{w}\in\mathbf{X}_N, \quad |a_N(\mathbf{u},\mathbf{w})| \leqslant c|\mathbf{u}|_{1,\Omega}|\mathbf{w}|_{1,\Omega}. \tag{40}$$

Proof. Thanks to the definition (20) of a_N, we must prove that, for any u and w in $\mathbf{X}_{N,x}$,

$$\left|\sum_{i=0}^{N}\sum_{j=1}^{N} (\partial^2 u/\partial x^2)(\xi_i,\zeta_j)w(\xi_i,\zeta_j)\rho_i\omega_j\right| \leqslant c\,\|\partial u/\partial x\|_{0,\Omega}\|\partial w/\partial x\|_{0,\Omega}, \tag{41}$$

$$\left|\sum_{i=0}^{N}\sum_{j=1}^{N}(\partial^2 u/\partial y^2)(\xi_i,\zeta_j)w(\xi_i,\zeta_j)\rho_i\omega_j\right| \leqslant c\,\|\partial u/\partial y\|_{0,\Omega}\|\partial w/\partial y\|_{0,\Omega}. \tag{42}$$

(1) First, due to formula (5), we have

$$\left|\sum_{i=0}^{N}\sum_{j=1}^{N}(\partial^2 u/\partial x^2)(\xi_i,\zeta_j)w(\xi_i,\zeta_j)\rho_i\omega_j\right|$$
$$=\left|\int_\Lambda \sum_{j=1}^{N}(\partial u/\partial x)(x,\zeta_j)(\partial w/\partial x)(x,\zeta_j)\omega_j\,dx\right|$$
$$\leqslant\left\{\int_\Lambda \sum_{j=1}^{N}(\partial u/\partial x)(x,\zeta_j)^2\omega_j\,dx\right\}^{1/2}\left\{\int_\Lambda \sum_{j=1}^{N}(\partial w/\partial x)(x,\zeta_j)^2\omega_j\,dx\right\}^{1/2}$$

Now, writing $(\partial u/\partial x)(x,y)=(1-y^2)r(x,y)$ and $(\partial w/\partial x)(x,y)=(1-y^2)t(x,y)$, where r and t belong to $\mathbb{P}_N(\bar\Lambda)\otimes\mathbb{P}_{N-1}(\bar\Lambda)$, and applying Lemma 2 yields (41).

(2) Next we set

$$u(x,y)=(1-y^2)\sum_{n=1}^{N}u_n(x)L_n'(y),\quad u_n\in\mathbb{P}_N(\bar\Lambda),$$
$$w(x,y)=(1-y^2)\sum_{n=1}^{N}w_n(x)L_n'(y),\quad w_n\in\mathbb{P}_N(\bar\Lambda)$$

and note that, due to (28),

$$(\partial^2 u/\partial y^2)(x,y)=-\sum_{n=1}^{N}u_n(x)n(n+1)L_n'(y).$$

Thanks to (4) and (5), we have

$$\left|\sum_{i=0}^{N}\sum_{j=1}^{N}(\partial^2 u/\partial y^2)(\xi_i,\zeta_j)w(\xi_i,\zeta_j)\rho_i\omega_j\right|$$
$$\leqslant\left|\sum_{i=0}^{N}\rho_i\left(\sum_{n=1}^{N-1}u_n(\xi_i)w_n(\xi_i)n^2(n+1)^2/(n+1/2)+u_N(\xi_i)w_N(\xi_i)\sum_{j=1}^{N}(1-\zeta_j^2)L_N'(\zeta_j)^2\omega_j\right)\right|.$$

Using Lemma 1 gives

$$\left|\sum_{i=0}^{N}\sum_{j=1}^{N}(\partial^2 u/\partial y^2)(\xi_i,\zeta_j)w(\xi_i,\zeta_j)\rho_i\omega_j\right|$$
$$\leqslant 2\left|\sum_{i=0}^{N}\rho_i\sum_{n=1}^{N}u_n(\xi_i)w_n(\xi_i)n^2(n+1)^2/(n+1/2)\right|$$
$$\leqslant 2\left\{\sum_{i=0}^{N}\rho_i\sum_{n=1}^{N}u_n(\xi_i)^2n^2(n+1)^2/(n+1/2)\right\}^{1/2}\left\{\sum_{i=0}^{N}\rho_i\sum_{n=1}^{N}w_n(\xi_i)^2n^2(n+1)^2/(n+1/2)\right\}^{1/2}$$

We recall (Lemma 3.2 of Reference 15) that formulae (5) and (6) yield for any φ in $\mathbb{P}_N(\bar{\Lambda})$

$$\int_\Lambda \varphi(\zeta)^2\,\mathrm{d}\zeta \leqslant \sum_{i=0}^{N} \varphi(\xi_i)^2\rho_i \leqslant 2\int_\Lambda \varphi(\zeta)^2\,\mathrm{d}\zeta. \tag{43}$$

Hence we obtain

$$\left|\sum_{i=0}^{N}\sum_{j=1}^{N} (\partial^2 u/\partial y^2)(\xi_i,\zeta_j)w(\xi_i,\zeta_j)\rho_i\omega_j\right|$$
$$\leqslant c\left\{\sum_{n=1}^{N}\|u_n\|_{0,\Lambda}^2 n^2(n+1)^2/(n+1/2)\right\}^{1/2}\left\{\sum_{n=1}^{N}\|z_n\|_{0,\Lambda}^2 n^2(n+1)^2/(n+1/2)\right\}^{1/2},$$

which is equivalent to (42).

Proposition 3

The form a_N satisfies the following condition of ellipticity:

$$\forall \mathbf{u}\in\mathbf{X}_N,\quad a_N(\mathbf{u},\mathbf{u})\geqslant cN^{-1}|\mathbf{u}|_{1,\Omega}^2. \tag{44}$$

Proof. Thanks to the definition (20) of a_N, it suffices to prove that, for any u in $\mathbf{X}_{N,x}$,

$$-\sum_{i=0}^{N}\sum_{j=1}^{N} (\partial^2 u/\partial x^2)(\xi_i,\zeta_j)u(\xi_i,\zeta_j)\rho_i\omega_j \geqslant cN^{-1}\|\partial u/\partial x\|_{0,\Omega}^2, \tag{45}$$

$$-\sum_{i=0}^{N}\sum_{j=1}^{N} (\partial^2 u/\partial y^2)(\xi_i,\zeta_j)u(\xi_i,\zeta_j)\rho_i\omega_j \geqslant c\|\partial u/\partial y\|_{0,\Omega}^2. \tag{46}$$

(1) Due to (5), we have

$$-\sum_{i=0}^{N}\sum_{j=1}^{N} (\partial^2 u/\partial x^2)(\xi_i,\zeta_j)u(\xi_i,\zeta_j)\rho_i\omega_j = \int_\Lambda \sum_{j=1}^{N} (\partial u/\partial x)(x,\zeta_j)^2\omega_j\,\mathrm{d}x.$$

Writing $(\partial u/\partial x)(x,y)=(1-y^2)r(x,y)$, where r belongs to $\mathbb{P}_{N-1}(\bar{\Lambda})\otimes\mathbb{P}_{N-1}(\bar{\Lambda})$, and using Lemma 2, we obtain immediately (45).

(2) We set

$$u(x,y)=(1-y^2)\sum_{n=1}^{N} u_n(x)L_n'(y),\quad u_n\in\mathbb{P}_N(\bar{\Lambda}),$$

so that

$$(\partial^2 u/\partial y^2)(x,y) = -\sum_{n=1}^{N} u_n(x)n(n+1)L_n'(y).$$

Due to (4) and (5), we have

$$-\sum_{i=0}^{N}\sum_{j=1}^{N} (\partial^2 u/\partial y^2)(\xi_i,\zeta_j)u(\xi_i,\zeta_j)\rho_i\omega_j$$
$$=\sum_{i=0}^{N}\rho_i\left(\sum_{n=1}^{N} u_n(\xi_i)^2 n(n+1)\sum_{j=1}^{N}(1-\zeta_j^2)L_n'(\zeta_j)^2\omega_j\right).$$

Using Lemma 1, then (28) and (43) gives

$$-\sum_{i=0}^{N}\sum_{j=1}^{N}(\partial^2u/\partial y^2)(\xi_i,\zeta_j)u(\xi_i,\zeta_j)\rho_i\omega_j$$
$$\geqslant \sum_{n=1}^{N}(n^2(n+1)^2/(n+1/2))\sum_{i=0}^{N}u_n(\xi_i)^2\rho_i$$
$$\geqslant \sum_{n=1}^{N}\|u_n\|_{0,\Lambda}^2 n^2(n+1)^2/(n+1/2)=\|\partial u/\partial y\|_{0,\Omega}^2,$$

whence (46).

Remark 5

The constant cN^{-1} in formula (44) is optimal. Indeed, we consider the polynomial $\mathbf{u}=(u,0)$, where u is given by

$$u(x,y)=s(x)(1-y^2)L_N''(y) \tag{47}$$

and s is a polynomial of $\mathbb{P}_N(\bar{\Lambda})\cap \mathbf{H}_0^1(\Lambda)$ satisfying

$$c_1\leqslant \|s\|_{0,\Lambda}+N^{-2}\|s'\|_{0,\Lambda}\leqslant c_2 \tag{48}$$

(then we have $a_N(\mathbf{u},\mathbf{u})\leqslant cN^6$ and $|\mathbf{u}|_{1,\Omega}^2\geqslant c'N^7$. An example of polynomial which satisfies (48) is

$$s(\zeta)=N^{-1}(1-\zeta^2)\sum_{n=2}^{N-2}s_n((2n+1)/n(n+1))L_n'(\zeta)$$

with s_n equal to n^2 if n is $\leqslant N/2$ and equal to $(N-n)^2$ if n is $>N/2$).

4. PROPERTIES OF THE DISCRETE PRESSURE GRADIENT

It remains now to study the properties of the bilinear form b_N defined in (21), which is related to the discretization of the pressure gradient in equation (1). We can already state:

Proposition 4

The form b_N is uniformly continuous with respect to N, i.e.,

$$\forall \mathbf{u}\in\mathbf{X}_N,\forall q\in\mathbf{M}_N,\quad |b_N(\mathbf{u},q)|\leqslant c|\mathbf{u}|_{1,\Omega}\|q\|_{0,\Omega}. \tag{49}$$

Proof. Thanks to the property (5) applied to b_N (see (22)), we must only prove that, for any u in $\mathbf{X}_{N,x}$ and q in $\mathbf{M}_N$,

$$\left|\int_\Lambda \mathrm{d}x\sum_{j=1}^{N}(\partial u/\partial x)(x,\zeta_j)q(x,\zeta_j)\omega_j\right|\leqslant c\|\partial u/\partial x\|_{0,\Omega}\|q\|_{0,\Omega}. \tag{50}$$

We set

$$(\partial u/\partial x)(x,y)=\sum_{n=0}^{N+1}r_n(x)L_n(y),\quad r_n\in\mathbb{P}_{N-1}(\bar{\Lambda}),$$
$$q(x,y)=\sum_{n=0}^{N-1}q_n(x)L_n(y),\quad q_n\in\mathbb{P}_{N-1}(\bar{\Lambda}),$$

so that, using (4) once more,

$$\left|\int_\Lambda \mathrm{d}x \sum_{j=1}^{N} (\partial u/\partial x)(x,\zeta_j)q(x,\zeta_j)\omega_j\right| = \left|\int_\Lambda \mathrm{d}x\left(\sum_{n=0}^{N-1} r_n(x)q_n(x)/(n+1/2)\right.\right.$$
$$\left.\left.+\, r_{N+1}(x)q_{N-1}(x)\sum_{j=1}^{N} L_{N+1}(\zeta_j)L_{N-1}(\zeta_j)\omega_j\right)\right|.$$

Since the ζ_j are the zeros of L_N, we have by (32)

$$\sum_{j=1}^{N} L_{N+1}(\zeta_j)L_{N-1}(\zeta_j)\omega_j = -(N/(N+1))\sum_{j=1}^{N} L_{N-1}(\zeta_j)^2\omega_j = -N/(N+1)(N-1/2).$$

We obtain

$$\left|\int_\Lambda \mathrm{d}x \sum_{j=1}^{N} (\partial u/\partial x)(x,\zeta_j)q(x,\zeta_j)\omega_j\right|$$
$$\leqslant \left\{\sum_{n=0}^{N-1} \|r_n\|_{0,\Lambda}^2/(n+1/2)\right\}^{1/2}\left\{\sum_{n=0}^{N-1} \|q_n\|_{0,\Lambda}^2/(n+1/2)\right\}^{1/2}$$
$$+\,\|r_{N+1}\|_{0,\Lambda}\|q_{N-1}\|_{0,\Lambda}N/(N+1)(N-1/2),$$

which yields (50).

The following proposition ensures that the space $\mathbf{M}_N$ contains no spurious mode, i.e., no function q such that $b_N(\mathbf{u},q)$ vanishes for any $\mathbf{u}$ in $\mathbf{X}_N$. This fact proves the compatibility between the spaces $\mathbf{X}_N$ and $\mathbf{M}_N$.

Proposition 5

The form b_N satisfies the following inf–sup condition:

$$\forall q\in\mathbf{M}_N, \exists\mathbf{u}\in\mathbf{X}_N, \mathbf{u}\neq\mathbf{0}/b_N(\mathbf{u},q)\geqslant cN^{-5/2}|\mathbf{u}|_{1,\Omega}\|q\|_{0,\Omega}. \tag{51}$$

Proof. Let q be any function in $\mathbf{M}_N$. We set

$$q=\sum_{m=0}^{N-1}\sum_{n=0}^{N-1} q_{mn}L_m(x)L_n(y), \quad \text{with } q_{00}=0.$$

Then we define $\mathbf{u}=(u,v)$ in $\mathbf{X}_N$ by

$$\forall x\in\bar{\Lambda}, \quad \begin{cases} u(x,\zeta_j) = -\displaystyle\sum_{m=1}^{N-1}\sum_{n=0}^{N-1} q_{mn}L_n(\zeta_j)(L_{m+1}(x)-L_{m-1}(x))/(2m+1), & 1\leqslant j\leqslant N, \\ u(x,\pm 1)=0, \end{cases} \tag{52}$$

and

$$\forall y\in\bar{\Lambda}, \quad \begin{cases} v(\zeta_i,y) = -\displaystyle\sum_{m=0}^{N-1}\sum_{n=1}^{N-1} q_{mn}L_m(\zeta_i)(L_{n+1}(y)-L_{n-1}(y))/(2n+1), & 1\leqslant i\leqslant N, \\ v(\pm 1,y)=0, \end{cases} \tag{53}$$

so that, due to (31),

$$(\partial u/\partial x)(x,\zeta_j) = -\sum_{m=1}^{N-1}\sum_{n=0}^{N-1} q_{mn}L_m(x)L_n(\zeta_j), \quad 1\leqslant j\leqslant N,$$

$$(\partial v/\partial y)(\zeta_j,y) = -\sum_{m=0}^{N-1}\sum_{n=1}^{N-1} q_{mn}L_m(\zeta_i)L_n(y), \quad 1\leqslant i\leqslant N.$$

Thanks to the property (5) applied to b_N, recalling that q_{00} is equal to 0, we have

$$\begin{aligned} b_N(\mathbf{u},q) &= \sum_{m=1}^{N-1}\sum_{n=0}^{N-1} q_{mn}^2\int_\Lambda \mathrm{d}x \sum_{j=1}^{N} L_m(x)^2L_n(\zeta_j)^2\omega_j \\ &\quad + \sum_{m=0}^{N-1}\sum_{n=1}^{N-1} q_{mn}^2\int_\Lambda \mathrm{d}y \sum_{i=1}^{N} L_m(\zeta_i)^2L_n(y)^2\omega_i \\ &\geqslant \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} q_{mn}^2/(m+1/2)(n+1/2), \end{aligned}$$

which means

$$b_N(\mathbf{u},q)\geqslant \|q\|_{0,\Omega}^2. \tag{54}$$

Moreover, writing $(\partial u/\partial x)(x,y)=(1-y^2)r(x,y)$, we note that

$$\|\partial u/\partial x\|_{0,\Omega}^2 = \int_\Lambda \mathrm{d}x\int_\Lambda (1-y^2)^2 r(x,y)^2\,\mathrm{d}y;$$

Lemma 2 implies

$$\begin{aligned} \|\partial u/\partial x\|_{0,\Omega}^2 &\leqslant cN\int_\Lambda \mathrm{d}x\sum_{j=1}^{N}(\partial u/\partial x)(x,\zeta_j)^2\omega_j \\ &\leqslant cN\sum_{m=1}^{N-1}\sum_{n=0}^{N-1} q_{mn}^2\int_\Lambda \mathrm{d}x\sum_{j=1}^{N} L_m^2(x)L_n^2(\zeta_j)\omega_j \leqslant c'N\|q\|_{0,\Omega}^2. \end{aligned}$$

Next we use the Poincaré–Friedrichs inequality

$$\|u\|_{0,\Omega}\leqslant c\|\partial u/\partial x\|_{0,\Omega}\leqslant c'N^{1/2}\|q\|_{0,\Omega},$$

then an inverse inequality (Lemma 2.4 of Reference 15), so that

$$\|\partial u/\partial y\|_{0,\Omega}\leqslant cN^2\|u\|_{0,\Omega}\leqslant cN^{5/2}\|q\|_{0,\Omega}.$$

Finally we obtain

$$|u|_{1,\Omega}\leqslant cN^{5/2}\|q\|_{0,\Omega}, \tag{55}$$

and, in exactly the same way,

$$|v|_{1,\Omega}\leqslant cN^{5/2}\|q\|_{0,\Omega}. \tag{56}$$

We deduce the lemma from (54)–(56).

5. ANALYSIS OF THE DISCRETE PROBLEM

We begin by stating that the discrete problem (11), (12) is well posed, since it is now a straightforward consequence of the variational formulation (25) together with Propositions 2–5 and a classical result about saddle-point problems.[9]

Theorem 1

For any $\mathbf{f}$ *in* $\mathscr{C}^0(\bar{\Omega})^2$, *problem (11), (12) has a unique solution* $(\mathbf{u}_N, p_N)$ *in* $\mathbf{X}_N \times \mathbf{M}_N$.

The main result of this section will consist of some error estimates between the solutions $(\mathbf{u}, p)$ and $(\mathbf{u}_N, p_N)$ of problems (1)–(3) and (11), (12) respectively. First we estimate the approximation error between a divergence-free function and its projection onto a subspace of divergence-free polynomials, following here an idea of Sacchi Landriani and the second author. More general results have been proved by Sacchi Landriani and Vandeven.[16]

Lemma 3

Let σ *be a real number* $\geqslant 1$. *For any function* $\mathbf{u}$ *in* $\mathbf{X} \cap H^\sigma(\Omega)^2$ *satisfying* $\operatorname{div} \mathbf{u} = 0$ *in* Ω, *there exists a function* $\tilde{\mathbf{u}}_N$ *in*

$$\{[\mathbb{P}_{N-1}(\bar{\Lambda}) \otimes \mathbb{P}_{N-2}(\bar{\Lambda})] \cap \mathbf{H}_0^1(\Omega)\} \times \{[\mathbb{P}_{N-2}(\bar{\Lambda}) \otimes \mathbb{P}_{N-1}(\bar{\Lambda})] \cap \mathbf{H}_0^1(\Omega)\},$$

satisfying $\operatorname{div} \tilde{\mathbf{u}}_N = 0$ *in* Ω, *such that*

$$|\mathbf{u}, \tilde{\mathbf{u}}_N|_{1,\Omega} \leqslant c N^{1-\sigma} \|\mathbf{u}\|_{\sigma,\Omega}. \tag{57}$$

Proof. Since $\mathbf{u}$ satisfies $\operatorname{div} \mathbf{u} = 0$ in Ω, we know that there exists a function ψ in $\mathbf{H}_0^2(\Omega)$ such that $\mathbf{u} = \operatorname{curl} \psi$ in Ω; moreover, ψ belongs to $\mathbf{H}^{\sigma+1}(\Omega)$. Let ψ_N denote the projection of ψ onto $\mathbb{P}_{N-1}(\bar{\Omega}) \cap \mathbf{H}_0^2(\Omega)$ with respect to the scalar product of $\mathbf{H}_0^2(\Omega)$. Clearly, if we set $\tilde{\mathbf{u}}_N = \operatorname{curl} \psi_N$, $\tilde{\mathbf{u}}_N$ belongs to

$$\{[\mathbb{P}_{N-1}(\bar{\Lambda}) \otimes \mathbb{P}_{N-2}(\bar{\Lambda})] \cap \mathbf{H}_0^1(\Omega)\} \times \{[\mathbb{P}_{N-2}(\bar{\Lambda}) \otimes \mathbb{P}_{N-1}(\bar{\Lambda})] \cap \mathbf{H}_0^1(\Omega)\}$$

and satisfies $\operatorname{div} \tilde{\mathbf{u}}_N = 0$ in Ω. Moreover, we have (Chapter II of Reference 17)

$$|\mathbf{u} - \tilde{\mathbf{u}}_N|_{1,\Omega} \leqslant c|\psi - \psi_N|_{2,\Omega} \leqslant c' N^{2-(\sigma+1)} \|\psi\|_{\sigma+1,\Omega} \leqslant c'' N^{1-\sigma} \|\mathbf{u}\|_{\sigma,\Omega}.$$

We need also to compare the discrete scalar product with the continuous one.

Lemma 4

Let ρ *be a real number* > 1. *For any function* $\mathbf{f}$ *in* $\mathbf{H}^\rho(\Omega)^2$, *the following estimate holds*:

$$\sup_{\mathbf{v}_N \in \mathbf{X}_N} \frac{(\mathbf{f}, \mathbf{v}_N) - (\mathbf{f}, \mathbf{v}_N)_N}{\|\mathbf{v}_N\|_{0,\Omega}} \leqslant c N^{1-\rho} \|\mathbf{f}\|_{\rho,\Omega}. \tag{58}$$

Proof. First let us introduce the interpolation operators $\imath_N$ and $\jmath_N$ associated with the Gauss and Gauss–Lobatto points respectively: for any φ in $\mathscr{C}^0(\bar{\Lambda})$, $\imath_N \varphi$ belongs to $\mathbb{P}_{N-1}(\bar{\Lambda})$ and satisfies

$$\forall i, 1 \leqslant i \leqslant N, \quad \imath_N \varphi(\zeta_i) = \varphi(\zeta_i), \tag{59}$$

while $\jmath_N \varphi$ belongs to $\mathbb{P}_N(\bar{\Lambda})$ and satisfies

$$\forall i, 0 \leqslant i \leqslant N, \quad \jmath_N \varphi(\xi_i) = \varphi(\xi_i). \tag{60}$$

Then, for any $\mathbf{f} = (f, g)$ in $\mathscr{C}^0(\bar{\Omega})^2$, the polynomial $\mathcal{J}_N \mathbf{f} = ((\jmath_N \otimes \imath_N) f, \ (\imath_N \otimes \jmath_N) g)$ is in $\{\mathbb{P}_N(\bar{\Lambda}) \otimes \mathbb{P}_{N-1}(\bar{\Lambda})\} \times \{\mathbb{P}_{N-1}(\bar{\Lambda}) \otimes \mathbb{P}_N(\bar{\Lambda})\}$. Moreover, we notice that

$$\|f - (\jmath_N \otimes \imath_N) f\|_{0,\Omega} \leqslant \|f - (\jmath_N \otimes \mathrm{id}) f\|_{0,\Omega} + \|f - (\mathrm{id} \otimes \imath_N) f\|_{0,\Omega} + \|(\mathrm{id} - \jmath_N) \otimes (\mathrm{id} - \imath_N) f\|_{0,\Omega}.$$

It is well known (Theorem 3.2 of Reference 15) that, for any φ in $\mathbf{H}^\sigma(\Lambda)$, $\sigma > 1/2$,

$$\|\varphi - \imath_N \varphi\|_{0,\Lambda} \leqslant c N^{1/2-\sigma} \|\varphi\|_{\sigma,\Omega}, \tag{61}$$

$$\|\varphi - \jmath_N \varphi\|_{0,\Lambda} \leqslant c N^{1/2-\sigma} \|\varphi\|_{\sigma,\Omega}. \tag{62}$$

Hence we obtain

$$\begin{aligned} \|f - (\jmath_N \otimes \imath_N) f\|_{0,\Omega} \leqslant\; & c(N^{1/2-\rho} \|f/f\|_{\mathbf{H}^\rho(\Lambda;\mathbf{L}^2(\Lambda))} + N^{1/2-\rho} \|f/f\|_{\mathbf{L}^2(\Lambda;\mathbf{H}^\rho(\Lambda))} \\ & + N^{(1-\rho)/2} \|f - (\mathrm{id} \otimes \imath_N) f\|_{\mathbf{H}^{\rho/2}(\Lambda;\mathbf{L}^2(\Lambda))}) \\ \leqslant\; & c(N^{1/2-\rho} \|f/f\|_{\mathbf{H}^\rho(\Lambda;\mathbf{L}^2(\Lambda))} + N^{1/2-\rho} \|f/f\|_{\mathbf{L}^2(\Lambda;\mathbf{H}^\rho(\Lambda))} \\ & + N^{(1-\rho)/2} N^{(1-\rho)/2} \|f\|_{\mathbf{H}^{\rho/2}(\Lambda;\mathbf{H}^{\rho/2}(\Lambda))}); \end{aligned}$$

Using Proposition 4.2.3 of Reference 18 together with an interpolation argument, we know that $\mathbf{H}^\rho(\Omega)$ is continuously imbedded into

$$\mathbf{H}^\rho(\Lambda; \mathbf{L}^2(\Lambda)) \cap \mathbf{L}^2(\Lambda; \mathbf{H}^\rho(\Lambda)) \cap \mathbf{H}^{\rho/2}(\Lambda; \mathbf{H}^{\rho/2}(\Lambda)),$$

whence

$$\|f - (\jmath_N \otimes \imath_N) f\|_{0,\Omega} \leqslant c N^{1-\rho} \|f\|_{\rho,\Omega}.$$

Estimating the term $\|g - (\imath_N \otimes \jmath_N) g\|_{0,\Omega}$ in a similar way, we finally derive

$$\|\mathbf{f} - \mathscr{I}_N \mathbf{f}\|_{0,\Omega} \leqslant c N^{1-\rho} \|\mathbf{f}\|_{\rho,\Omega}. \tag{63}$$

It is well known (Theorem 2.3 of Reference 15) that there exists a polynomial $\mathbf{f}_N$ in

$$\{\mathbb{P}_{N-1}(\bar{\Lambda}) \otimes \mathbb{P}_{N-2}(\bar{\Lambda})\} \times \{\mathbb{P}_{N-2}(\bar{\Lambda}) \otimes \mathbb{P}_{N-1}(\bar{\Lambda})\},$$

such that

$$\|\mathbf{f} - \mathbf{f}_N\|_{0,\Omega} \leqslant c N^{-\rho} \|\mathbf{f}\|_{\rho,\Omega}. \tag{64}$$

Clearly, due to (4) and (5), we have for any $\mathbf{v}_N$ in $\mathbf{X}_N$

$$(\mathbf{f}, \mathbf{v}_N) - (\mathbf{f}, \mathbf{v}_N)_N = (\mathbf{f} - \mathbf{f}_N, \mathbf{v}_N) + (\mathbf{f}_N - \mathscr{I}_N \mathbf{f}, \mathbf{v}_N)_N.$$

Using a Cauchy–Schwarz inequality, then (43) and Lemma 2 yields

$$\begin{aligned} (\mathbf{f}, \mathbf{v}_N) - (\mathbf{f}, \mathbf{v}_N)_N &\leqslant \|\mathbf{f} - \mathbf{f}_N\|_{0,\Omega} \|\mathbf{v}_N\|_{0,\Omega} + c \|\mathbf{f}_N - \mathscr{I}_N \mathbf{f}\|_{0,\Omega} \|\mathbf{v}_N\|_{0,\Omega} \\ &\leqslant c(\|\mathbf{f} - \mathbf{f}_N\|_{0,\Omega} + \|\mathbf{f} - \mathscr{I}_N \mathbf{f}\|_{0,\Omega}) \|\mathbf{v}_N\|_{0,\Omega}; \end{aligned}$$

this last inequality, together with (63) and (64), gives the result.

We are now in position to compare the continuous and the discrete problems. This is stated in the following two theorems.

Theorem 2

If the solution $(\mathbf{u}, p)$ *of problem (1)–(3) belongs to* $\mathbf{H}^\sigma(\Omega)^2 \times \mathbf{H}^{\sigma-1}(\Omega)$ *for a real number* $\sigma \geqslant 1$ *and the data* $\mathbf{f}$ *belong to* $\mathbf{H}^\rho(\Omega)^2$ *for a real number* $\rho > 1$, *the following error estimate holds*:

$$|\mathbf{u} - \mathbf{u}_N|_{1,\Omega} \leqslant c\{N^{2-\sigma}(\|\mathbf{u}\|_{\sigma,\Omega} + \|p\|_{\sigma-1,\Omega}) + N^{2-\rho}\|\mathbf{f}\|_{\rho,\Omega}\}. \tag{65}$$

Proof. First let $\mathbf{v}_N$ be any element of $\mathbf{X}_N$ such that

$$\forall q_N \in \mathbf{M}_N, \quad b_N(\mathbf{v}_N, q_N) = 0.$$

From (25), using Proposition 3, we derive

$$|\mathbf{u}_N-\mathbf{v}_N|_{1,\Omega}^2 \leqslant cNa_N(\mathbf{u}_N-\mathbf{v}_N,\mathbf{u}_N-\mathbf{v}_N)=cN\{-a_N(\mathbf{v}_N,\mathbf{u}_N-\mathbf{v}_N)+(\mathbf{f},\mathbf{u}_N-\mathbf{v}_N)_N\}.$$

Thanks to (19), we have

$$|\mathbf{u}_N-\mathbf{v}_N|_{1,\Omega}^2 \leqslant cN\{a(\mathbf{u},\mathbf{u}_N-\mathbf{v}_N)-a_N(\mathbf{v}_N,\mathbf{u}_N-\mathbf{v}_N)+b(\mathbf{u}_N-\mathbf{v}_N,p)$$
$$-(\mathbf{f},\mathbf{u}_N-\mathbf{v}_N)+(\mathbf{f},\mathbf{u}_N-\mathbf{v}_N)_N\},$$

so that, for any q_N in $\mathbf{M}_N$,

$$|\mathbf{u}_N-\mathbf{v}_N|_{1,\Omega}^2 \leqslant cN\{a(\mathbf{u}-\mathbf{v}_N,\mathbf{u}_N-\mathbf{v}_N)+(a-a_N)(\mathbf{v}_N,\mathbf{u}_N-\mathbf{v}_N)+b(\mathbf{u}_N-\mathbf{v}_N,p-q_N)$$
$$+(b-b_N)(\mathbf{u}_N-\mathbf{v}_N,q_N)-(\mathbf{f},\mathbf{u}_N-\mathbf{v}_N)+(\mathbf{f},\mathbf{u}_N-\mathbf{v}_N)_N\}.$$

Next, the properties (4) and (5) of the quadrature formulae yield that

$$\forall \mathbf{w}_N\in\mathbf{X}_N, \forall q_N\in\mathbb{P}_{N-2}(\bar{\Omega}),\quad b(\mathbf{w}_N,q_N)=b_N(\mathbf{w}_N,q_N). \tag{66}$$

Moreover, we note that the function $\tilde{\mathbf{u}}_N$ defined in Lemma 3, since it belongs to

$$\{\mathbb{P}_{N-1}(\bar{\Lambda})\otimes\mathbb{P}_{N-2}(\bar{\Lambda})\}\times\{\mathbb{P}_{N-2}(\bar{\Lambda})\otimes\mathbb{P}_{N-1}(\bar{\Lambda})\},$$

satisfies

$$\forall q_N\in\mathbf{M}_N,\quad b_N(\tilde{\mathbf{u}}_N,q_N)=b(\tilde{\mathbf{u}}_N,q_N)=-\int_\Omega \operatorname{div}\tilde{\mathbf{u}}_N q_N\,\mathrm{d}\mathbf{x}=0,$$

and also

$$a(\tilde{\mathbf{u}}_N,\mathbf{u}_N-\tilde{\mathbf{u}}_N)=a_N(\tilde{\mathbf{u}}_N,\mathbf{u}_N-\tilde{\mathbf{u}}_N).$$

Then, choosing $\mathbf{v}_N=\tilde{\mathbf{u}}_N$, we obtain, for any q_N in $\mathbb{P}_{N-2}(\bar{\Omega})$,

$$|\mathbf{u}_N-\tilde{\mathbf{u}}_N|_{1,\Omega}^2 \leqslant cN\{a(\mathbf{u}-\tilde{\mathbf{u}}_N,\mathbf{u}_N-\tilde{\mathbf{u}}_N)+b(\mathbf{u}_N-\tilde{\mathbf{u}}_N,p-q_N)-(\mathbf{f},\mathbf{u}_N-\tilde{\mathbf{u}}_N)+(\mathbf{f},\mathbf{u}_N-\tilde{\mathbf{u}}_N)_N\}.$$

From the continuity of a and b, we derive

$$|\mathbf{u}_N-\tilde{\mathbf{u}}_N|_{1,\Omega}\leqslant cN\left\{|\mathbf{u}-\tilde{\mathbf{u}}_N|_{1,\Omega}+\|p-q_N\|_{0,\Omega}+\sup_{\mathbf{v}_N\in\mathbf{X}_N}\frac{(\mathbf{f},\mathbf{v}_N)-(\mathbf{f},\mathbf{v}_N)_N}{\|\mathbf{v}_N\|_{0,\Omega}}\right\},$$

so that

$$|\mathbf{u}-\mathbf{u}_N|_{1,\Omega}\leqslant cN\left\{|\mathbf{u}-\tilde{\mathbf{u}}_N|_{1,\Omega}+\|p-q_N\|_{0,\Omega}+\sup_{\mathbf{v}_N\in\mathbf{X}_N}\frac{(\mathbf{f},\mathbf{v}_N)-(\mathbf{f},\mathbf{v}_N)_N}{\|\mathbf{v}_N\|_{0,\Omega}}\right\}. \tag{67}$$

It is well known (Theorem 2.3 of Reference 15) that

$$\inf_{q_N\in\mathbb{P}_{N-2}(\bar{\Omega})}\|p-q_N\|_{0,\Omega}\leqslant cN^{1-\sigma}\|p\|_{\sigma-1,\Omega}. \tag{68}$$

Using this estimate and Lemma 4, together with (67), gives (65).

Remark 6

We recall (Theorem 3.3.3.1 of Reference 19 and Theorem II.2 of Reference 20) that, for any $\rho\leqslant\rho_0\simeq 1{\cdot}7396$, the solution $(\mathbf{u},p)$ belongs to $\mathbf{H}^{\rho+2}(\Omega)^2\times\mathbf{H}^{\rho+1}(\Omega)$ whenever $\mathbf{f}$ is in $\mathbf{H}^\rho(\Omega)^2$. Hence, for $\mathbf{f}$ smooth enough, the discrete velocity $\mathbf{u}_N$ always converges towards the exact one.

Theorem 3

If the solution $(\mathbf{u},p)$ of problem (*1*)–(*3*) belongs to $\mathbf{H}^\sigma(\Omega)^2\times\mathbf{H}^{\sigma-1}(\Omega)$ *for a real number* $\sigma\geqslant 1$ *and* *the data* $\mathbf{f}$ *belong to* $\mathbf{H}^{\rho}(\Omega)^2$ *for a real number* $\rho > 1$, *the following error estimate holds*:

$$\|p - p_N\|_{0,\Omega} \leqslant cN^{5/2}\{N^{2-\sigma}(\|u\|_{\sigma,\Omega} + \|p\|_{\sigma-1,\Omega}) + N^{2-\rho}\|\mathbf{f}\|_{\rho,\Omega}\}. \tag{69}$$

Proof. Let q_N be any element in $\mathbf{M}_N$. Using Proposition 5, we have

$$\|p_N - q_N\|_{0,\Omega} \leqslant cN^{5/2} \sup_{\mathbf{v}_N \in \mathbf{X}_N} \frac{b_N(\mathbf{v}_N, p_N - q_N)}{|\mathbf{v}_N|_{1,\Omega}}.$$

Due to (25) and (19), we compute for any $\mathbf{v}_N$ in $\mathbf{X}_N$

$$\begin{aligned} b_N(\mathbf{v}_N, p_N - q_N) &= -a_N(\mathbf{u}_N, \mathbf{v}_N) - b_N(\mathbf{v}_N, q_N) + (\mathbf{f}, \mathbf{v}_N)_N \\ &= a(\mathbf{u} - \mathbf{u}_N, \mathbf{v}_N) + (a - a_N)(\mathbf{u}_N, \mathbf{v}_N) \\ &\quad + b(\mathbf{v}_N, p - q_N) + (b - b_N)(\mathbf{v}_N, q_N) - (\mathbf{f}, \mathbf{v}_N) + (\mathbf{f}, \mathbf{v}_N)_N. \end{aligned}$$

Next we note that

$$(a - a_N)(\mathbf{u}_N, \mathbf{v}_N) = (a - a_N)(\mathbf{u}_N - \tilde{\mathbf{u}}_N, \mathbf{v}_N) + (a - a_N)(\tilde{\mathbf{u}}_N, \mathbf{v}_N) = (a - a_N)(\mathbf{u}_N - \tilde{\mathbf{u}}_N, \mathbf{v}_N);$$

from the continuity of a, a_N and b, together with (4), we obtain, for any q_N in $\mathbb{P}_{N-2}(\bar{\Omega})$,

$$\|p - p_N\|_{0,\Omega} \leqslant cN^{5/2}\left\{|\mathbf{u} - \mathbf{u}_N|_{1,\Omega} + |\mathbf{u} - \tilde{\mathbf{u}}_N|_{1,\Omega} + \|p - q_N\|_{0,\Omega} + \sup_{\mathbf{v}_N \in \mathbf{X}_N} \frac{(\mathbf{f}, \mathbf{v}_N) - (\mathbf{f}, \mathbf{v}_N)_N}{\|\mathbf{v}_N\|_{0,\Omega}}\right\}. \tag{70}$$

Using (65), (68) and Lemmas 3 and 4 in (70) gives the proposition.

Remark 7

A collocation pseudospectral problem similar to (11), (12) can be formulated to discretize the full Navier–Stokes equations. By using the same arguments as for periodic non-periodic boundary conditions[6] together with the previous results, we can also study this non-linear problem and, if $(\mathbf{u}, p)$ is a non-singular and sufficiently smooth solution of the Navier–Stokes equations, we derive exactly the same error estimates (65) and (69) as in the linear case.

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